

From Elasticity to Hypoplasticity: Dynamics of Granular Solids

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Abstract

“Granular elasticity,” useful for calculating static stress distributions in granular media, is generalized by including the effects of slowly moving, deformed grains. The result is a hydrodynamic theory for granular solids that agrees well with models from soil mechanics.

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Granular media has different phases that, in dependence of the grain's ratio of deformation to kinetic energy, may loosely be referred to as gaseous, liquid and solid. The first phase is relatively well understood: Moving fast and being free most of the time, the grains in the gaseous phase have much kinetic, but next to none elastic, energy [1]. In the denser liquid phase, say in chute flows, there is less kinetic energy, more deformation, and a rich rheology that has been scrutinized recently [2]. In granular statics, with the grains deformed but stationary, the energy is all elastic. This state is legitimately referred to as solid because static shear stresses are sustained. If granular solid is slowly sheared, the predominant part of the energy remains elastic. Yet no theory is capable of accounting for both its statics and dynamics, and no picture exists that helps to render its physics transparent.

Two grains in contact are initially very compliant, because so little material is being deformed. As this geometric fact should also hold on larger scales, for many grains, *diverging compliance at diminishing compression* is a basic characteristics of granular solids, and the reason it is sensible to abandon the approximation of infinitely rigid grains. Starting from this observation, a theory termed GE (for “granular elasticity”) was constructed to account for static granular stress distributions. Taking the energy w as a function of u_{ij} , the elastic contribution to the total strain field ε_{ij} , we specify [3]

$$w = \sqrt{\Delta} \left(\mathcal{B} \frac{2}{5} \Delta^2 + \mathcal{A} u_s^2 \right) = \mathcal{B} \sqrt{\Delta} \left(\frac{2}{5} \Delta^2 + u_s^2 / \xi \right), \quad (1)$$

with $\Delta \equiv -u_{\ell\ell}$, $u_s^2 \equiv u_{ij}^0 u_{ij}^0$, $u_{ij}^0 \equiv u_{ij} - \frac{1}{3} u_{\ell\ell} \delta_{ij}$. (The notations: $a_{ij}^0 \equiv a_{ij} - \frac{1}{3} a_{\ell\ell} \delta_{ij}$ and $a_s^2 \equiv a_{ij}^0 a_{ij}^0$ with any a_{ij} are employed throughout this paper.) The elastic coefficient \mathcal{B} , a measure of overall rigidity, is a function of the density. Denoting ρ_g as the granular material's bulk density, and $e \equiv \rho_g / \rho - 1$ as the void ratio, we take $\mathcal{B} = \mathcal{B}_0 \times (2.17 - e)^2 / [1.3736(1 + e)]$, with $\mathcal{B}_0, \xi > 0$ two material constants. The elastic energy w contributes $\pi_{ij} \equiv -\partial w / \partial u_{ij}$ to the total stress σ_{ij} . And since the elastic stress is the only contribution in statics, force balance reads $\nabla_j \sigma_{ij} = \nabla_j \pi_{ij} = \rho G_i$. This was solved for three classical cases: silos, sand piles and granular sheets under a point load, resulting in rather satisfactory agreement to experiments, see [4]. Moreover, the energy w (with $P \equiv \frac{1}{3} \pi_{\ell\ell}$) is convex only for $\pi_s / P \leq \sqrt{2/\xi}$, implying no elastic solution is stable beyond it. Identifying this as the yield surface gives $\xi \approx 5/3$ for natural sand.

When granular solid is being slowly sheared, we must expect a qualitative change of its behavior: In addition to moving with the large-scaled velocity v_i , the grains also move and

slip in deviation of it – implying a small but finite granular temperature T_g . As a result, some of the grains are temporarily unjammed, with enough time to decrease their deformation. This depletes the elastic energy and relaxes the static stress. Stress relaxation is typical of viscoelastic systems such as polymers. Granular media are similar, but they possess a relaxation rate that vanishes with T_g . This is the reason they return to perfect elasticity when stationary. The basic physics of granular solids, *viscoelasticity at finite T_g* , is in fact epitomized by a sand pile, which holds its shape when unperturbed, but fails to do so when tapped. A set of differential equations termed *granular solid hydrodynamics* (GSH) is derived consistently below starting from GE, with this simple physics as the only additional input.

Conservation of density and momentum always holds,

$$\frac{\partial}{\partial t}\rho + \nabla_i(\rho v_i) = 0, \quad \frac{\partial}{\partial t}(\rho v_i) + \nabla_j(\sigma_{ij} + \rho v_i v_j) = \rho G_i, \quad (2)$$

where G_i is the gravitational constant. In granular gas or liquid, the stress σ_{ij} has the same structure as in the Navier-Stokes equation, though the viscosity is a function of the shear. In granular solid, the stress is not usually taken to be given in a closed form. Instead, constitutive relations are employed. These relate the temporal derivatives of stress and strain, giving $\frac{\partial}{\partial t}\sigma_{ij}$ as a function of $v_{ij} \equiv \frac{1}{2}(\nabla_i v_j + \nabla_j v_i)$ and density (where $\frac{\partial}{\partial t}$ is often replaced by an objective derivative say from Jaumann).

Hypoplasticity, or HPM (for hypoplastic model), is a modern, well-verified, yet comparatively simple theory of soil mechanics [5]. It is quite realistic in the above specified regime of solid dynamics, though less appropriate for determining static stress distributions. The starting point is the rate-independent constitutive relation,

$$\frac{\partial}{\partial t}\sigma_{ij} = H_{ijkl}v_{kl} + \Lambda_{ij}\sqrt{v_{\ell k}^0 v_{\ell k}^0 + \epsilon (v_{\ell\ell})^2}, \quad (3)$$

where the coefficients $H_{ijkl}, \Lambda_{ij}, \epsilon$ are functions of σ_{ij}, ρ , specified using experimental data mainly from triaxial apparatus. Great efforts are invested in finding accurate expressions for them, of which a recent set [5] is $\epsilon = 1/3$,

$$H_{ijkl} = f \left(F^2 \delta_{ik} \delta_{jl} + a^2 \sigma_{ij} \sigma_{kl} / \sigma_{nn}^2 \right), \quad (4)$$

$$\Lambda_{ij} = a f f_d F \left(\sigma_{ij} + \sigma_{ij}^0 \right) / \sigma_{nn}, \quad (5)$$

where [with $a = 2.76$, $h_s = 1600$ MPa, $e_d = 0.44e_i$, $e_c = 0.85e_i$, $e_i^{-1} = \exp(\sigma_{\ell\ell}/h_s)^{0.19}$, e the

void ratio]

$$f_d = \left(\frac{e - e_d}{e_c - e_d} \right)^{0.25}, \quad f = -\frac{8.7h_s(1 + e_i)}{3(\sigma_s/\sigma_{\ell\ell} + 1)e} \left(\frac{\sigma_{\ell\ell}}{h_s} \right)^{0.81},$$

$$F = \sqrt{\frac{3\sigma_s^2}{8\sigma_{\ell\ell}^2} + \frac{2\sigma_s^2\sigma_{\ell\ell} - 3\sigma_s^4/\sigma_{\ell\ell}}{2\sigma_s^2\sigma_{\ell\ell} - 6\sigma_{ij}^0\sigma_{j\ell}^0\sigma_{\ell i}^0}} - \sqrt{\frac{3}{8}} \frac{\sigma_s}{\sigma_{\ell\ell}}.$$

If GSH as derived below from the idea given above reduces to HPM under certain conditions, we would have, on one hand, captured valuable insights into the physics of this field-tested theory, understood its range of validity, how to widen it by appropriate modifications, and on the other hand, obtained a broadside verification of GSH, along with the physical picture embedded in it. As we shall see, GSH indeed reduces to Eq (3) for a stationary T_g , with H_{ijkl} , Λ_{ij} , ϵ given in terms of $M_{ijk\ell} \equiv -\partial^2 w / \partial u_{ij} \partial u_{k\ell}$ (known from GE) and four new scalars [combinations of transport coefficients such as viscosities and stress relaxation rates, see Eq (17)]. Although quite different from Eqs (4,5), the new H_{ijkl} , Λ_{ij} , ϵ yield very similar accounts in all cases we have considered.

A large part of GSH may be duplicated from the hydrodynamic theory of transient elasticity, constructed to describe polymers [6]. This theory accounts for any system in which both the elastic energy and stress relax, irrespective how this happens microscopically – whether due to polymer strands disentangling, or the grains unjamming. (A formal and rather more detailed derivation of GSH can be found in an accompanying paper [7].) The stress σ_{ij} and the elastic strain u_{ij} are determined by

$$\sigma_{ij} = \pi_{ij} - \sigma_{ij}^D, \quad \left(\frac{\partial}{\partial t} + v_k \nabla_k \right) u_{ij} = v_{ij} + X_{ij}, \quad (6)$$

where $\pi_{ij} \equiv -\partial w / \partial u_{ij}$ is the elastic stress and $v_{ij} \equiv \frac{1}{2}(\nabla_i v_j + \nabla_j v_i)$. σ_{ij}^D and X_{ij} are the irreversible contributions, given by Onsager relations that connect the “currents,” σ_{ij}^D , X_{ij} , to the “forces,” v_{ij} , π_{ij} ,

$$\sigma_{ij}^D = (\eta + \eta_g) v_{ij}^0 + (\zeta + \zeta_g) \delta_{ij} v_{\ell\ell} + \alpha \pi_{ij}, \quad (7)$$

$$X_{ij} = -\alpha v_{ij} + \beta \pi_{ij}^0 + \beta_1 \delta_{ij} \pi_{\ell\ell} \quad (8)$$

$$= -\alpha v_{ij} - \frac{1}{\tau} u_{ij}^0 - \frac{1}{\tau_1} \delta_{ij} u_{\ell\ell}. \quad (9)$$

The coefficients $\eta, \zeta, \eta_g, \zeta_g > 0$ in σ_{ij}^D are viscosities, see below for their differences. Calculating $\frac{\partial}{\partial t} \sigma_{ij}$ as in Eq (3), they all vanish for steady velocities, $\frac{\partial}{\partial t} v_i = 0$. The term X_{ij} , accounting for the relaxation of the elastic strain u_{ij} , is rather more consequential. Eq (9) is obtained

by taking the derivative of Eq (1), $\pi_{ij} \equiv -\partial w/\partial u_{ij} = \sqrt{\Delta}(\mathcal{B}\Delta\delta_{ij} - 2\mathcal{A}u_{ij}^0) + \mathcal{A}(u_s^2/2\sqrt{\Delta})\delta_{ij}$. So the relaxation times are given as $1/\tau \equiv 2\beta\mathcal{A}\sqrt{\Delta}$, $1/\tau_1 \equiv 3\beta_1\sqrt{\Delta}(\mathcal{B} + \frac{1}{2}\mathcal{A}u_s^2/\Delta^2)$. The coefficient α is a cross coefficient of the Onsager matrix. It is taken as a scalar for simplicity.

In principle, the transport coefficients η , η_g , ζ , ζ_g , τ , τ_1 , α are functions of the thermodynamic variables: density, temperature and the elastic strain u_{ij} . We shall, again for simplicity, assume that they are strain-independent, while noting three points: (1) Constant τ, τ_1 implies strain-dependent β, β_1 . Choosing the former as constant and not the latter, the trace and traceless part of $\frac{\partial}{\partial t}u_{ij}$ are decoupled. (2) As discussed above, $1/\tau, 1/\tau_1$ vanish with T_g . So the obvious and simplest assumption is

$$1/\tau = \lambda T_g, \quad 1/\tau_1 = \lambda_1 T_g, \quad (10)$$

with $\lambda, \lambda_1, \tau_1/\tau = \lambda/\lambda_1$ possibly functions of the density, but independent from stress and T_g . (3) Being reactive, α is not restricted in its magnitude. It may stay constant while $1/\tau, 1/\tau_1$ vary – though it must eventually vanish for $1/\tau, 1/\tau_1 \rightarrow 0$, as $\alpha = 0$ in statics.

The above hydrodynamic theory is closed if we amend it with an equation of motion for T_g . In thermodynamics, the energy change dw from all microscopic, implicit variables is subsumed as Tds , with s the entropy and $T \equiv \partial w/\partial s$ its conjugate variable. From this, we divide out the kinetic energy of granular random motion, executed by the grains in deviation from the ordered, large-scale motion, and denote it as $T_g ds_g$, calling s_g , $T_g \equiv \partial w/\partial s_g$ granular entropy and temperature. In other words, we consider two heat reservoirs, the first containing the energy of granular random motion, the second the rest of all microscopic degrees of freedom, especially phonons. In equilibrium, $T_g = T$, and s_g is part of s . But when the granular system is being tapped or sheared, and T_g is many orders of magnitude larger than T , then this leaky, intermediary heat reservoir produces physics in its own right. Taking s_g as the part of the entropy accounting for the granular kinetic energy, our definition is fairly close to the entropy of granular gas [1], though its functional dependence is probably dominated by the effect of excluded volumes. The entropy s , on the other hand, is closer to the so-called “configurational entropy,” [8] (see section 6 of the first of [4] for a discussion of their relationship). The balance equations are $\frac{\partial}{\partial t}s + \nabla_k(sv_k) = R/T$, $\frac{\partial}{\partial t}s_g + \nabla_k(s_g v_k) = R_g/T_g$, where

$$R = \eta v_s^2 + \zeta v_{\ell\ell}^2 + \beta \pi_s^2 + \beta_1 \pi_{\ell\ell}^2 + \gamma T_g^2, \quad (11)$$

$$R_g = \eta_g v_s^2 + \zeta_g v_{\ell\ell}^2 - \gamma T_g^2. \quad (12)$$

The first four terms in the entropy production R are the usual contributions from shear flow and stress relaxation, as given by transient elasticity. The first two terms of R_g account analogously for shear excitation of random motion. The term γT_g^2 (with $\gamma > 0$) describes how the kinetic energy of random motion seeps from s_g into s . (Diffusion of T, T_g are easily included when needed.)

With Eqs (1, 2, 6, 7, 9, 10, 12, 11), GSH is complete. It especially contains the equilibrium case, $\sigma_{ij} = \pi_{ij}$, in which the dissipative fields vanish, $\sigma_{ij}^D, X_{ij} = 0$. Off equilibrium, these two fields are finite, and we calculate $\frac{\partial}{\partial t}\sigma_{ij}$ assuming $\frac{\partial}{\partial t}v_i = 0$, from Eqs (6, 7, 9),

$$\begin{aligned} \frac{\partial}{\partial t}\sigma_{ij} &= (1 - \alpha)\frac{\partial}{\partial t}\pi_{ij} = (1 - \alpha)M_{ijkl}\frac{\partial}{\partial t}u_{kl} = \\ &= (1 - \alpha)M_{ijkl}[(1 - \alpha)v_{kl} - \frac{1}{\tau}u_{kl}^0 - \frac{1}{\tau_1}\delta_{kl}u_{\ell\ell}]. \end{aligned} \quad (13)$$

As mentioned above, the energy w loses its convexity at $\pi_s/P = \sqrt{6/5}$, and no static, elastic solution is possible beyond this ratio. Therefore, it was identified as yield. Given Eq (13), the same identification holds dynamically: The loss of convexity implies that one of the six eigenvalues of $M_{ijkl} \equiv -\partial^2 w / \partial u_{ij} \partial u_{kl}$ (written as a 6×6 matrix) vanishes at this point, and a strain rate along the associated direction yields vanishing stress rate.

For $R_g = 0$, when s_g is being produced and leaking at the same rate, we have a stationary T_g , given as

$$T_g = \sqrt{\eta_g/\gamma} \sqrt{v_s^2 + (\zeta_g/\eta_g)v_{\ell\ell}^2}. \quad (14)$$

Inserting Eqs (10,14) into (13), we retrieve Eq (3), with

$$H_{ijkl} = (1 - \alpha)^2 M_{ijkl}, \quad \epsilon = \zeta_g/\eta_g, \quad (15)$$

$$\Lambda_{ij} = (1 - \alpha)M_{ijkl}[(\tau/\tau_1)\Delta\delta_{kl} - u_{kl}^0]\lambda\sqrt{\eta_g/\gamma}. \quad (16)$$

HPM has 43 free parameters (36+6+1 for $H_{ijkl}, \Lambda_{ij}, \epsilon$), all functions of the stress and density. Expressed as here, the stress and density dependence are essentially determined by M_{ijkl} that (with $\xi = 5/3$ and $\mathcal{B}_0 = 8500$ MPa) is a known quantity [4]. For the four free constants, we take

$$1 - \alpha = 0.22, \quad \frac{\tau}{\tau_1} = 0.09, \quad \frac{\zeta_g}{\eta_g} = 0.33, \quad \lambda\sqrt{\frac{\eta_g}{\gamma}} = 114, \quad (17)$$

to be realistic choices, as these numbers yield satisfactory agreement with HPM. Their significance are: $\zeta_g/\eta_g = 0.33$ implies shear flows are three times as effective in creating T_g as compressional flows. $\tau/\tau_1 = 0.09$ means, plausibly, that the relaxation rate of shear stress

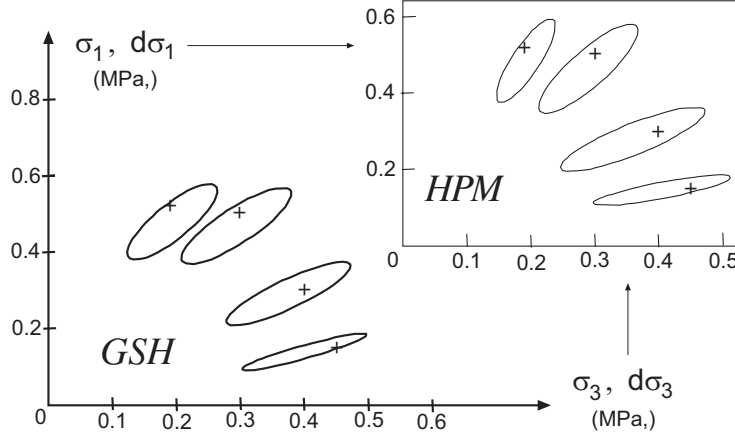


FIG. 1: The stress changes $d\sigma_1, d\sigma_3$, calculated using GSH (granular solid hydrodynamics) and HPM (hypoplastic model), for given strain rate starting from different points (depicted as crosses) in the stress space spanned by σ_1, σ_3 . The strain rate has varying directions but a constant amplitude, $\sqrt{2v_1^2 + v_3^2}$, such that the applied strain changes form circles around each cross (not shown).

is ten times higher than that of pressure. For a purely elastic system, Eq (3) is replaced by $\frac{\partial}{\partial t}\sigma_{ij} = M_{ijkl}v_{lk}$. Therefore, the factor $(1 - \alpha)^2$ accounts for an overall, dynamic softening of the static compliance tensor M_{ijkl} , a known effect in soil mechanics [9]. Finally, λ controls the stress relaxation rate for given T_g , and $\sqrt{\eta_g/\gamma}$ how well shear flow excites T_g . Together, $\lambda\sqrt{\eta_g/\gamma} = 114$ determines the relative weight of plastic versus reactive response. (Note $|\Lambda_{ij}|/|H_{ijkl}| \sim |u_{kl}^0| \cdot 114/(1 - \alpha)$ is, for $|u_{ij}^0|$ around 10^{-3} , of order unity.)

Next, we compare Eqs (15, 16) to (4, 5) in their results with respect to “response envelopes,” a standard test in soil mechanics for rating constitutive relations [5]. Axial symmetry of the triaxial geometry is assumed, with σ_{ij}, v_{ij} diagonal, and $\sigma_1 \equiv \sigma_{xx} = \sigma_{yy}$, $\sigma_3 \equiv \sigma_{zz}$, $v_1 \equiv v_{xx} = v_{yy}$, $v_3 \equiv v_{zz}$, $P \equiv \frac{2}{3}\sigma_1 + \frac{1}{3}\sigma_3$, $q \equiv \sigma_3 - \sigma_1$, $\sigma_s^2 \equiv \frac{2}{3}q^2$, $d\gamma \equiv (v_1 - v_3)dt$, $d\varepsilon \equiv -(2v_1 + v_3)dt$. Starting from a point in the stress space (spanned by σ_1, σ_3 in Fig 1 and σ_s, P in Fig 2), one deforms the system for a constant time dt , at given strain or stress rates, while recording the change in the conjugate quantity. Varying the direction, the input is a circle around the starting point, but the response envelopes show deformation characteristic

of the system, or the constitutive relation to be rated. Fig 1 and 2 show respectively the responding stress and strain envelopes, for the void ratio $e = 0.66$, calculated using GSH and HPM. The similarity in stress-dependence and anisotropy is obvious.

In Fig 3, one strain envelope is blown up for a more detailed comparison, using the extended version of response envelope as given in [10]. Here, the applied stress rate is reversed at halftime, such that the system returns to the starting point in stress space at the end. The responding strain change, depicted as deflected, straight dotted lines, does not return to the origin. Both GSH and HPM predict that the end points from all angles of stress changes (some of the angles are given at the deflection points) form a straight line OA. (Instead of a line, a narrow ellipse is reported in the 2D-simulation of [10]. This may be a result of the fact that the stationarity of T_g is briefly violated when the stress rate is reversed, during which the system is rather less plastic.) OA's angle σ in strain space is usually referred to as the “flow direction,” while the direction in stress space, along which the plastic deformation is largest (with the strain starting at O and ending at A) is called

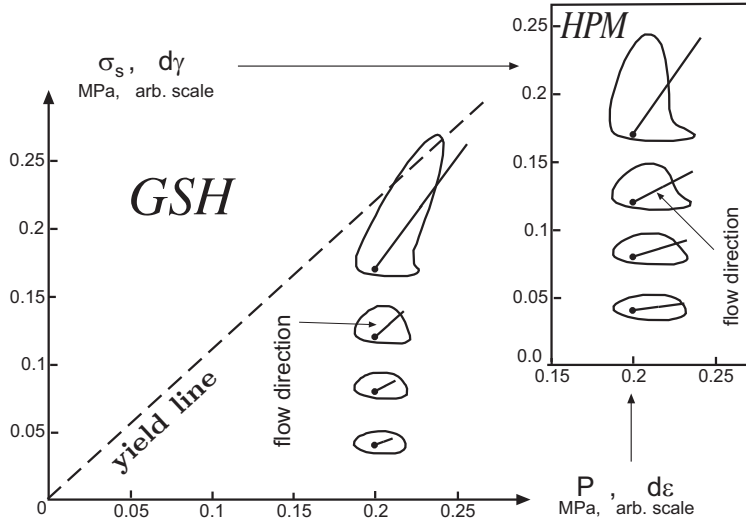


FIG. 2: The change in strain $d\gamma, d\epsilon$ for given stress rate starting from different points in the stress space, spanned by σ_s, P . The amplitude of the stress rate $\sqrt{dP^2 + dq^2}$ is constant. See Fig 3 for an explanation of the “flow direction.”

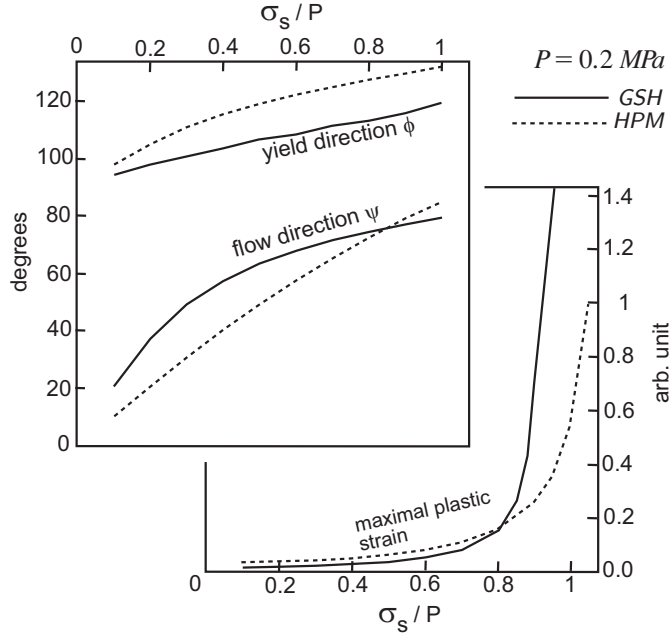


FIG. 4: Yield direction, flow direction, and the maximal plastic strain (length of OA), versus σ_s/P , for $P = 0.2$ MPa, calculated employing GSH and HPM, respectively.

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- [1] P. K. Haff, J. Fluid Mech., **134**, 401(1983); J. T. Jenkins and S. B. Savage, J. Fluid Mech., **130**, 187(1983).
 - [2] L.E. Silbert, D. Ertas, G.S. Grest, T.C. Halsey, D. Levine, S.J. Plimpton, Phys. Rev. **E 64**, 051302 (2001); GDR MiDi group, Eur. Phys. J. **E 14**, 341 (2004); P.Jop, Y. Forterre, O. Pouliquen, Nature **441**, 727, 2006.
 - [3] Y.M. Jiang, M. Liu, Phys. Rev. Lett., **91**, 144301 (2003), **93**, 148001(2004); Eur. Phys. J. E., 1292-8941(2007).
 - [4] D.O. Krimer, M. Pfitzner, K. Bräuer, Y. Jiang, M. Liu, Phys. Rev. **E74**, 061310 (2006); K. Bräuer, M. Pfitzner, D.O. Krimer, M. Mayer, Y. Jiang, M. Liu, Phys. Rev. **E74**, 061311 (2006)
 - [5] D. Kolymbas, *Introduction to Hypoplasticity*, (Balkema, Rotterdam, 2000); W. Wu & D. Kolymbas, in *Constitutive Modelling of Granular Materials* (ed Kolymbas, Springer-Verlag, Berlin, 2000), and references therein.

- [6] H. Temmen, H. Pleiner, M. Liu, H.R. Brand, Phys. Rev. Lett. **84**, 3228 (2000); H. Pleiner, M. Liu, H.R. Brand, Acta Rheol. **43**, 502 (2004). (Nonlinear convective terms such as $u_{ki}v_{jk}$ or $u_{ki}\pi_{jk}$ are not displayed, because granular media typically consist of hard grains, with $u_{ij} \ll 1$. So these terms are negligible when compared to v_{jk} and π_{jk} . The total strain ε_{ij} , of course, is usually quite large.)
- [7] Y.M. Jiang, M. Liu, Cond-Mat, arXiv:0706.1352
- [8] S.F. Edwards, R.B.S. Oakeshott, Physica A 157, 1080 (1989); A. Metha, S.F. Edwards, Physica A 157, 1091.
- [9] A. Niemunis and I. Herle, Mech. of Cohes.-Frict. Mater., **2**, 279 (1997). It is perhaps useful to note that a softening effect may also be achieved by a T_g -dependence of the elastic coefficient B, with $\xi = 5/3$ unchanged.
- [10] F. Alonso-Marroquin and H. J. Herrmann, Phys. Rev. E **66**, 021301(2002) and Phys. Rev. Lett., **92**, 054301(2004); F. Alonso-Marroquin, S. Luding, H. J. Herrmann, I. Vardoulakis, Phys. Rev. E **71**, 051304(2005).